Ten Years of Interconnection and Damping Assignment Passivity–based Control of Mechanical Systems

Romeo Ortega

Laboratoire des Signaux et Systèmes, CNRS–Supelec

France

Starting Point:

R. Ortega, M. Spong, F. Gomez and G. Blankenstein: Stabilization of underactuated mechanical systems via interconnection and damping assignment, *IEEE Trans. Automatic Control*, Vol. AC–47, No. 8, August 2002, pp. 1218–1233.

Contents

1. IDA–PBC of Mechanical Systems
2. Constructive Solution for Under–actuation Degree One. (J. Acosta, R. Ortega, A. Astolfi and A. Mahindrakar: IDA–PBC of mechanical systems with underactuation degree one, *IEEE Trans. Automatic Control*, Vol 50, No. 12, 2005.)
3. Simplifying the PDEs via Coordinate Changes. (G. Viola, R. Ortega, R. Banavar, J. A. Acosta and A. Astolfi: Total energy shaping control of mechanical systems: simplifying the matching equations via coordinate changes, *IEEE Trans. Automatic Control*, Vol. 52, No. 6, pp. 1093–1099, 2007.)
4. Robustness to External Disturbances. (R. Ortega and J. Romero: Robust integral control of port-Hamiltonian systems: The case of non-passive outputs with unmatched disturbances, *Systems and Control Letters*, Vol. 61, No. 1, pp. 11-17, 2012.)
5. Globally Exponentially Stable Output Feedback Tracking. (J. Romero and R. Ortega: A globally exponentially stable tracking controller for mechanical systems using position feedback, *2013 American Control Conference*, June 17–19, 2013, Washington DC, USA.)
6. LTI Systems. (Z. Liu, R. Ortega and H. Su: Control via IDA of linear time–invariant systems: A tutorial, *International Journal of Control*, Vol 85, Issue 3, pp. 603-611, 2012.)

Supélec CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

1. IDA–PBC of Mechanical Systems

- Model

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u$$

where $H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + V(q)$, rank $(G) = m < n$.

- Desired closed–loop dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}(q)M_d(q) \\ -M_d(q)M^{-1}(q) & J_2(q,p) - G(q)K_v G^\top(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H_d}{\partial q} \\ \frac{\partial H_d}{\partial p} \end{bmatrix}$$

with $K_v > 0$ and $J_2(q,p) = -J_2^\top(q,p)$.

- Desired energy is parameterized
  - $H_d(q,p) = \frac{1}{2}p^\top M_d^{-1}(q)p + V_d(q)$, $M_d(q) = M_d^\top(q) > 0$
  - $q_\star = \arg\min V_d(q)$.
- Connection with Controlled Lagrangians: Gyroscopic terms had to be added

$$L_c(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q)M_d^{-1}(q)M(q)\dot{q} + \dot{q}^\top Q(q) - V_d(q)$$

Proposition

Let (without loss of generality)

$$J_2(q,p) = \frac{1}{2}\sum_{k=1}^{n} U_k(q)p_k,\; U_k = -U_k^\top.$$

Assume solution of the PDEs

$$\begin{aligned} G^\perp\{\frac{\partial^\top}{\partial q}(M^{-1}_{(\cdot,k)}) - M_d M^{-1}\frac{\partial^\top}{\partial q}(M_d^{-1})_{(\cdot,k)} + U_k M_d^{-1}\} &= 0 \\ G^\perp\{\frac{\partial V}{\partial q} - M_d M^{-1}\frac{\partial V_d}{\partial q}\} &= 0 \end{aligned}$$

with $(\cdot)_{(\cdot,k)}$ the k–th row, $G^\perp(q) \in \mathbb{R}^{(n-m)\times n}$ a full rank left annihilator of G, i.e., $G^\perp G = 0$ and $\text{rank}(G^\perp) = n - m$. Then, the system in closed–loop with

$$u = (G^\top G)^{-1}G^\top(\nabla_q H - M_d M^{-1}\nabla_q H_d + J_2 M_d^{-1}p) - K_v G^\top \nabla_p H_d,$$

takes the desired port–Hamiltonian form and $(q^\star, 0)$ is a stable equilibrium point with Lyapunov function H_d.

2. Constructive Solution For Underactuation Degree One

- Identification of a class of mechanical systems for which the PDEs are explicitly solved.

Proposition

- A.1 $m = n - 1$.
- A.2 M and V do not depend on the unactuated coordinate. The former can be enforced with Spong's partial feedback linearization.
- A.3 G and M are functions of a single element of q, say q_r, $r \in \{1, \ldots, n\}$

An explicit solution of the PDE's is given by

$$M_d(q_r) = \int_{q_r^\star}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M_d^0$$

$$V_d(q) = \int_0^{q_r} \frac{G^\perp \nabla V(\mu)}{\gamma_r(\mu)} d\mu + \Phi(z(q)),$$

where

$$\gamma := M^{-1} M_d (G^\perp)^\top, \; z(q) := q - \int_0^{q_r} \frac{\gamma(\mu)}{\gamma_r(\mu)} d\mu$$

and $\Psi = \Psi^\top$, $M_d^0 = (M_d^0)^\top > 0$ and Φ may be arbitrarily chosen.

3. Simplifying the PDEs via Coordinate Changes

- The KE-PDE is nonlinear and nonhomogeneous. The presence of the forcing term introduces a quadratic term in M_d that renders very difficult its solution—even with the help of the free skew–symmetric matrix J_2.
- Perform a coordinate change $(q,p) \hookrightarrow (q,\tilde{p})$, with $p = T(q)\tilde{p}$, where $T \in \mathbb{R}^{n\times n}$ is full rank. This yields:

$$\tilde{\Sigma} : \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & T^{-\top} \\ -T^{-1} & -T^{-1}(S - S^\top)T^{-\top} \end{bmatrix} \begin{bmatrix} \nabla_q \tilde{H} \\ \nabla_{\tilde{p}} \tilde{H} \end{bmatrix} + \begin{bmatrix} 0 \\ T^{-1}G \end{bmatrix} u,$$

where $\tilde{H}(q,\tilde{p}) = \frac{1}{2}\tilde{p}^\top T^\top(q) M^{-1}(q) T(q)\tilde{p} + V(q)$, and $S(q,\tilde{p}) = \nabla_q (T(q)\tilde{p})$.

- Define new target dynamics, in the coordinates $(q,\tilde{p})$, as

$$\tilde{\Sigma}_d : \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}(q)T(q)\tilde{M}_d(q) \\ -\tilde{M}_d T^\top(q)(q) M^{-1}(q) & \tilde{J}_2(q,\tilde{p}) \end{bmatrix} \begin{bmatrix} \nabla_q \tilde{H}_d \\ \nabla_{\tilde{p}} \tilde{H}_d \end{bmatrix},$$

where $\tilde{H}_d(q,\tilde{p}) = \frac{1}{2}\tilde{p}^\top \tilde{M}_d^{-1}(q)\tilde{p} + \tilde{V}_d(q)$ and $\tilde{J}_2 = -\tilde{J}_2^\top$ is free.

Obtaining an Homogeneous KE–PDE

Proposition T is such that

$$\sum_{i=1}^{n}\left[T^\top M^{-1}e_i G_k^\perp \frac{\partial T}{\partial q_i} + \frac{\partial T^\top}{\partial q_i}(e_i G_k^\perp)^\top M^{-1}T + G_k^\perp e_i T^\top \frac{\partial M^{-1}}{\partial q_i} T\right] = 0.$$

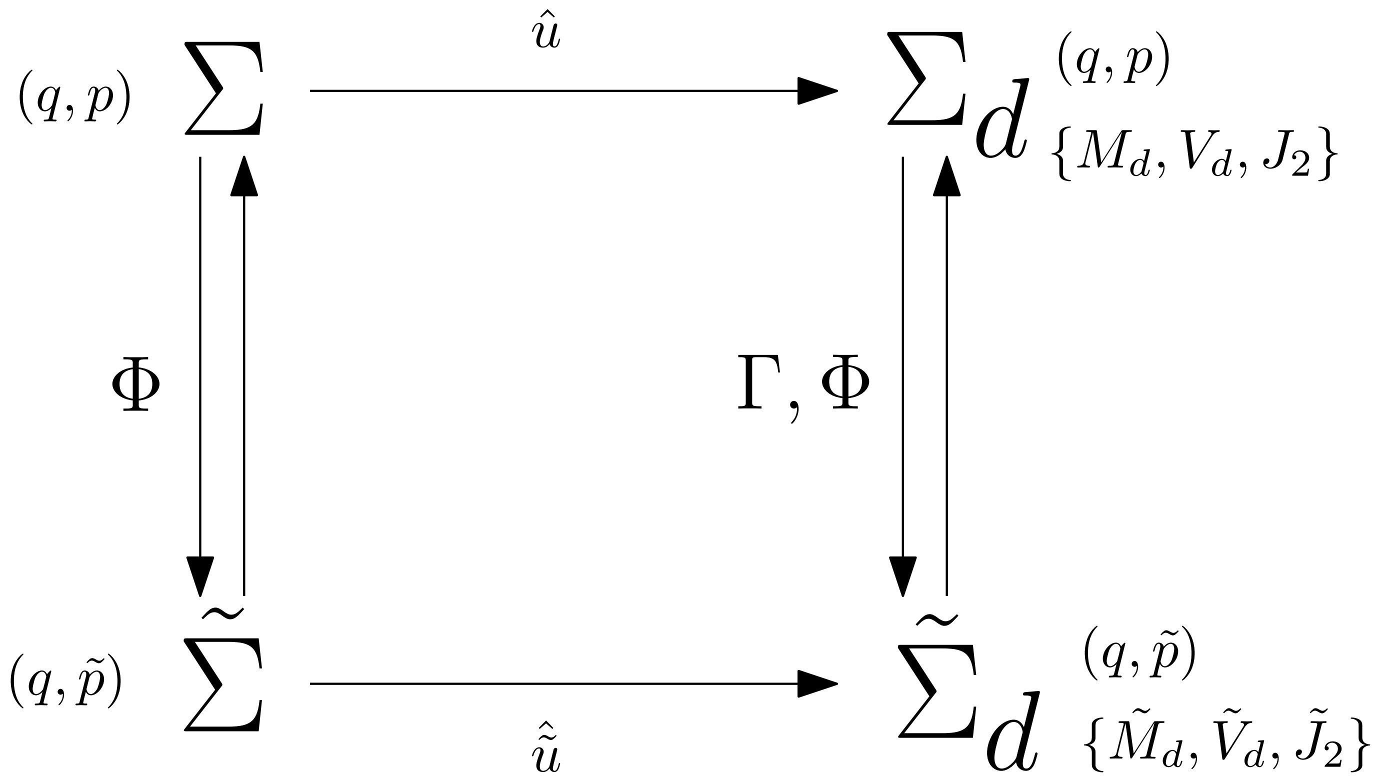
The PDEs become

$$\begin{aligned} G^\perp T\left[\tilde{M}_d T^\top M^{-1}\nabla_q(\tilde{p}^\top \tilde{M}_d^{-1}\tilde{p}) - 2\tilde{J}_2\tilde{M}_d^{-1}\tilde{p}\right] &= 0 \\ G^\perp T\tilde{M}_d T^\top M^{-1}\nabla\tilde{V}_d &= G^\perp \nabla V, \end{aligned}$$

Remarks

- $T = M$ solves the new PDE if and only if $G^\perp(q)C(q,\dot{q})\dot{q} = 0$, where $C \in \mathbb{R}^{n\times n}$ is the matrix of Coriolis and centrifugal forces of the system.
- Solving the new PDEs is, in principle, simpler: it has been possible for several practical examples, including the pendulum of Furuta ♡.
- For the Acrobot ♡ first proof of smooth stabilization with domain of attraction including the lower half plane.

Relationship Between New and Original Problem



$\Gamma : \{M_d, V_d, J_2\} \to \{\tilde{M}_d, \tilde{V}_d, \tilde{J}_2\}$ is one–to–one.

$\Phi : (q,p) \to (q,\tilde{p})$ is the coordinate transformation.

4. Robustness to External Disturbances

- Perturbed port–Hamiltonian (pH) model

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & -K_p \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ I_n \end{bmatrix} u + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

  with Hamiltonian

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q).$$

- d_1, d_2 are disturbances—possibly time–varying, but bounded.
- $K_p > 0$, $q^\star = \arg\min V(q) \quad \Rightarrow \quad$ Asymptotic stability if $d = 0$.
- Objective: Design a state–feedback controller that:
  - preserves asymptotic stability for constant disturbances,
  - ensures input–to–state stability (ISS).
- Main technical tools (Donaire/Junco, Automatica'09), (Ortega/Romero, SCL'10):
  - Change of coordinates (preserving pH structure and Hamiltonian function form)
  - Addition of integral action

Destabilization of Integral Action on Velocities

- Integral control on passive output

$$\begin{aligned} u &= -\eta \\ \dot{\eta} &= K_i M^{-1}(q)p, \quad K_i > 0 \end{aligned}$$

- If d_1 is a non–zero constant the system admits no constant equilibrium, and if $d_1 = 0$ and d_2 is constant there is an equilibrium set

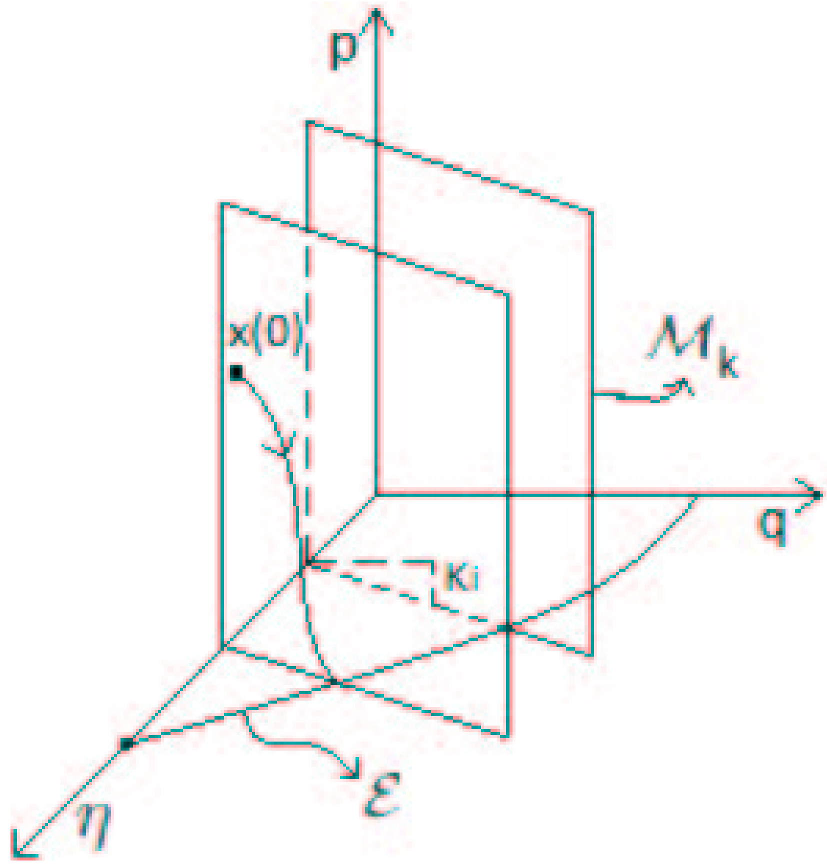
$$\mathcal{E} = \Big\{(q,p,\eta)\,|\,p = 0,\ \nabla V(q) + \eta = d_2\Big\}.$$

- With or without disturbances, the foliation

$$\mathcal{M}_\kappa = \Big\{(q,p,\eta)\,|\,K_i q - \eta = \kappa,\ \kappa \in \mathbb{R}\Big\},$$

  is invariant.
- Convergence to $(q^\star, 0, d_2)$ is attained only for a zero measure set of initial conditions.

Invariant Foliation in the State Space



Robustness for Constant Inertia Matrix and $d(t) = \bar{d}$

Proposition (Romero, et al., CDC'12) Constant inertia matrix M and disturbances $d = \bar{d}$ with PI control

$$\begin{aligned} u &= -K_p z_3 - M K_i \nabla V \\ \dot{z}_3 &= K_i \nabla V. \end{aligned}$$

(i) The closed–loop dynamics expressed in the coordinates,

$$z_1 = q, \quad z_2 = p + M(z_3 - K_p^{-1} d_2)$$

takes the pH form

$$\dot{z} = \begin{bmatrix} 0 & I_n & -K_i \\ -I_n & -K_p & 0 \\ K_i & 0 & 0 \end{bmatrix} \nabla H_z(z),$$

with

$$H_z(z) := H(z) + \frac{1}{2}(z_3 - z_3^*)^\top K_i^{-1}(z_3 - z_3^*).$$

(ii) $z^\star := (q^\star, 0, z_3^*)$, is asymptotically stable.

Non–constant $M(q)$: Change of Coordinates

Fact (Venkatraman, et al., TAC'10) Consider the system without damping $(K_p = 0)$ and no unmatched disturbances ($d_1 = 0$). The change of coordinates

$$(q, \bar{p}) = (q, T(q)p), \quad M^{-1}(q) = T^2(q).$$

transforms the dynamics into

$$\begin{bmatrix} \dot{q} \\ \dot{\bar{p}} \end{bmatrix} = \begin{bmatrix} 0 & T(q) \\ -T(q) & J_2(q,\bar{p}) \end{bmatrix} \nabla W + \begin{bmatrix} 0 \\ I_n \end{bmatrix} v + \begin{bmatrix} 0 \\ T d_2 \end{bmatrix},$$

with $v := T(q)u$, new Hamiltonian function

$$W(q,\bar{p}) = \frac{1}{2}|\bar{p}|^2 + V(q),$$

and the gyroscopic forces matrix

$$J_2(q,\bar{p}) := \nabla^\top (Tp)T - T\nabla(Tp)|_{p=T^{-1}\bar{p}}.$$

Robustness *vis–à–vis* $d_2(t)$

Proposition (Romero, et al., WLHM'12) Control law

$$\begin{aligned} v &= -(\nabla^2 V T + J_2 + R_2 + R_3)\bar{p} - (R_2 + R_3) z_3 - (T + R_2 + R_3)\nabla V \\ \dot{z}_3 &= (T + R_3)\nabla V + R_3 \bar{p} \end{aligned}$$

(i) Closed–loop dynamics in the coordinates $(z_1, z_2, z_3) = (q, \bar{p} + \nabla V(q) + z_3, z_3)$, is given by

$$\dot{z} = \begin{bmatrix} -T & T & -T \\ -T & -R_2 & -R_3 \\ T & R_3 & -R_3 \end{bmatrix} \nabla U + \begin{bmatrix} 0 \\ T d_2 \\ 0 \end{bmatrix}, \; U(z) := \frac{1}{2}|z_2|^2 + V(z_1) + \frac{1}{2}|z_3|^2$$

(ii) The closed–loop system is ISS (with respect to the disturbance $d_2(t)$).

(iii) If $d_2(t) = \bar{d}_2$, the equilibrium $z^\star = (q^*, 0, z_3^*)$ is asymptotically stable.

Remark Similar result for $(d_1(t), d_2(t))$, with complex control.

5. Globally Exponentially Stable Output Feedback Tracking

Proposition For all twice differentiable, bounded, references $(q_d(t), \mathbf{p}_d(t)) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists a dynamic position–feedback IDA–PBC that ensures uniform global exponential stability of the closed–loop system provided the inertia matrix is bounded from above. More precisely, there exist two (smooth) mappings

$$\mathbf{F} : \mathbb{R}^{3n+1} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{3n+1}, \; \mathbf{H} : \mathbb{R}^{3n+1} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$$

such that the mechanical system in closed–loop with

$$\dot{\chi} = \mathbf{F}(\chi, q, t), \; u = \mathbf{H}(\chi, q, t)$$

is a (perturbed) port–Hamiltonian system that verifies

$$\left| \begin{bmatrix} q(t) - q_d(t) \\ \mathbf{p}(t) - \mathbf{p}_d(t) \\ \chi(t) \end{bmatrix} \right| \leq \kappa \exp^{-\alpha(t-t_0)} \left| \begin{bmatrix} q(t_0) - q_d(t_0) \\ \mathbf{p}(t_0) - \mathbf{p}_d(t_0) \\ \chi(t_0) \end{bmatrix} \right|, \; \forall t \geq t_0.$$

for all initial conditions $(q(t_0), \mathbf{p}(t_0), \chi(t_0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{3n} \times \mathbb{R}_{\geq 0}$. Moreover, the controller ensures uniform global asymptotic stability even if the inertia matrix is not bounded from above.

6. Linear Time Invariant (Conservative) Mechanical Systems

- IDA for LTI systems: Find $u(x)$ such that

$$\dot{x} = Ax + Bu(x) \equiv F\nabla H_d$$

with $H_d(x) = \frac{1}{2}x^\top Px$, $P > 0$ and $F + F^\top \leq 0$.

- Proposition (Prajna, et al., SCL'02) IDA applicable if and only if (A, B) is stabilizable.
- IDA for mechanical systems: Given $H(q,p) = \frac{1}{2}|p|^2 + \frac{1}{2}q^\top Cq$ find $u(q,p)$ such that

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ G \end{bmatrix} u(q,p) \equiv \begin{bmatrix} 0 & M_d \\ -M_d & 0 \end{bmatrix} \nabla H_d.$$

where $H_d(q,p) = \frac{1}{2}p^\top M_d^{-1}p + \frac{1}{2}q^\top C_d q, \quad M_d > 0, C_d > 0.$

- Differences with general IDA is that H_d is separable and the structure of F is fixed.
- Proposition (Liu, et al., IJC'12), (Zenkov, MTNS'02), (Chang, SIAM/JOTA'10)
  - IDA is applicable if and only if the matrix associated to the uncontrollable part of the pair $(-C, G)$—if any—is diagonalizable and has negative real eigenvalues.
  - Stabilizability is not enough for applicability of IDA.

Thanks a lot Mark!